

Asymptotically Optimal Multi-Object Auctions

Electronic Market Design

- Re-consider economic mechanisms in view of their use in computational settings
 - Number of participants (to be discussed today)
 - Risk elements
 - Competition
- Computational aspects of mechanism design
 - Tractable multi-object auctions
 - Trading algorithms
 - Mechanism design for resource bounded agents
- Incorporating distributed systems features into game-theoretic models
 - Distributed games
 - Parallel games
 - Coordinators for cooperation

Auctions

Why auctions?

Auctions are the basic building blocks of protocols for non-cooperative environments, and the basic strategic mechanism in e-commerce applications.

Our aim: study and design optimal multi-object auctions for arbitrary participants, while incorporating features typical to electronic trades.

The economic environment: participants, goods, and allocations

- $N = \{1, 2, \dots, n\}$ – potential buyers/agents
- 0 – the seller/center
- G – a set of goods
- $\pi : G \rightarrow N \cup \{0\}$ – an allocation
- $\pi_i = \{a \in G : \pi(a) = i\}$
- Π – the set of possible allocations
- $\Delta(\Pi)$ – the set of all probability distributions on Π .

The economic environment: valuations and utilities

- $v_i : \Pi \rightarrow R$ – valuation function for agent i
- V_i – a compact set of feasible valuation functions for agent i .
- The valuation function $v_i \in V_i$ is determined by a random variable \tilde{v}_i
- The $(\tilde{v}_i)_{i \in N}$ are distributed on $V = V_1 \times V_2 \times \dots \times V_n$ according to the probability measure q
- $u_i : R \rightarrow R$ is a strictly increasing concave utility (for money) function of agent i , normalized with $u_i(0) = 0$.
- The utility of agent i , with valuation v_i , when the allocation is π and it is asked to pay p is $u_i(v_i(\pi) - p)$

Auction mechanisms

An auction mechanism: $((M_i)_{i \in N}, g, t)$, where:

- M_i – set of potential messages by agent i
- $M = M_1 \times M_2 \times \dots \times M_n$
- $g : M \rightarrow \Delta(\Pi)$ – the outcome function
- $t = (t_1, t_2, \dots, t_n)$
- $t_i : M \times \Pi \rightarrow R$ – the (monetary) transfer function for agent i

The above is general; dynamic auctions are analyzed by considering whole protocols as messages

The dynamics of an auction

- $m = (m_1, m_2, \dots, m_n)$ – sent from the agents to the center
- The center conducts a lottery according to the distribution $g(m)$, and selects the allocation π .
- Agent i gets π_i , and needs to transfer $t_i(m, \pi)$
- The utility of agent i is $u_i(v_i(\pi) - t_i(m, \pi))$
- An agent may decide not to participate, by sending a null message φ . In this case it will be assigned no goods and will pay 0.

Bayesian games and equilibrium

Auction mechanism (M, g, t) defines a Bayesian game:

- $b_i : V_i \rightarrow M_i$ – a strategy for agent i
- Σ_i – set of possible strategies for agent i
- $\Sigma = \Sigma_1 \times \Sigma_2 \times \dots \times \Sigma_n$ – strategy profiles
- If the agents use the vector of strategies $b = (b_1, b_2, \dots, b_n) \in \Sigma$, then the expected utility of i is denoted by $L_i(b)$, that is

$$L_i(b) = \int_V \sum_{\pi \in \Pi} u_i[v_i(\pi) - t_i(b(v), \pi)] g_\pi(b(v)) dq(v),$$

where $b(v) = (b_1(v_1), b_2(v_2), \dots, b_n(v_n))$.

- The valuation functions of the agents are their private information (but the whole setting is common-knowledge)
- A vector of strategies $b^e \in \Sigma$ is in equilibrium if for every $i \in N$, $\max_{b_i \in \Sigma_i} L_i(b_i, b_{-i}^e)$ is attained at b_i^e .

Basic Assumptions

A1 : No externalities ($v_i : 2^G \rightarrow R$)

A2 : Free disposal

A3 : The independent private value model [IPV]: $V_i = V_j$ for every i, j , and the random variables that determine the agents' valuation functions are independent and identically distributed (notice however that the agents may have different attitude to risk, i.e., different utility functions for money)

A4 : Risk-aversion (Risk-Neutrality is a specific case)

A Brief Example

- One good o , and 2 agents
- $V = V_1 = V_2 = [0, 1]$
- The agent's valuations are independently drawn from the uniform distribution on $[0, 1]$
- $u(y) = y$ is the utility functions of both agents
- A first-price (FP) auction:
 - $M_1 = M_2 = R_+$ – set of monetary bids
 - If $m = (m_1, m_2)$ then g assigns o to the agent $w(m) = \operatorname{argmax}_{i \in \{1, 2\}}(m_i)$, $t_{w(m)} = m_{w(m)}$, and the other agent pays nothing.

In a continuous symmetric increasing equilibrium we have that an agent needs to maximize his expected payoff (where x is the agent's valuation for the good, and z is the bid he makes):

$$\int_{v=0}^{b^{-1}(z)} (x - z) dv = b^{-1}(z)(x - z)$$

By taking the derivative and comparing to 0 at $z = b(x)$, we get:

$$-x + \frac{(x - (b(x)))}{b'(x)} = \frac{-xb'(x) + x - b(x)}{b'(x)} = 0$$

Hence, an equilibrium is $b(x) = \frac{x}{2}$

Maximizing Revenue

For a given auction mechanism and a given equilibrium b^e (of the corresponding Bayesian game), we denote by \tilde{R} the random revenue variable of the organizer:

$$\tilde{R}(v) = \sum_{i=1}^n \left(\sum_{\pi \in \Pi} t_i(b^e(v), \pi) g_{\pi}(b^e(v)) \right)$$

We denote the expected value of \tilde{R} by $E(\tilde{R})$. Given an economic environment (participants, goods, valuation functions, utility functions) we search for a mechanism that maximizes expected revenue.

Maximal Surplus

- \tilde{S} – the maximal-surplus random variable defined on $V = V_1 \times V_2 \cdots \times V_n$:

$$\tilde{S}(v) = \max_{\pi \in \Pi} \sum_{i=1}^n v_i(\pi_i)$$

- The expected value of \tilde{S} is denoted by $E(\tilde{S})$:

$$E(\tilde{S}) = \int_V \tilde{S}(v) dq(v)$$

If the allocation in an equilibrium of an auction always maximizes surplus, we say that this auction is (economically) efficient. An efficient auction needs not be revenue maximizing.

Basic results and questions

Most results have been obtained under [IPV]:

- Myerson solved the optimal auction problem for a single good and risk-neutral agents (with linear utility functions).
- Krishna and Perry discuss optimal multi-object auctions where agents are risk-neutral, and only efficient auctions are considered
- $FP > SP$ when agents are risk-averse, i.e. have concave utility functions (Riley and Samuelson, Maskin and Riley)
- We have shown that $TP > SP > FP$ for risk-seeking agents (with convex utility functions).

1. What is an optimal auction when selling several objects to arbitrary risk-averse agents?
2. Can we bound the revenue obtained by arbitrary auctions (given that agents overbid in equilibrium)?

These are questions of fundamental importance due to the flexibility and configurability of Internet Auctions.

An upper bound

We prove:

Assume A1, A2, and A4. Then the expected revenue of the seller in equilibrium is bounded above by the expected maximal surplus. That is,

$$E(\tilde{R}) \leq E(\tilde{S}).$$

Can we match the upper bound by an appropriate mechanism?

We are interested in exploring properties of electronic transactions, such as that we have many participants.

The Clarke Mechanism

- $M_i = V_i \cup \{\varphi\}$.
- $N(m)$ – the set of all active agents (for which $m_i \neq \varphi$)
- $g : M \rightarrow \Pi$:
 - for $m = (m_1, m_2, \dots, m_n)$ with $N(m) \neq \emptyset$, $g(m)$ is an allocation that maximizes $\sum_{i \in N(m)} v_i(\pi_i)$ over $\pi \in \Pi_{N(m)}$.
 - If $N(m) = \emptyset$ then $g(m)(a) = 0$ for every $a \in G$
- The transfer function of i , $t_i(m) = t_i(m, g(m))$:
 - If $N(m) = \{i\}$, then $t_i(m) = 0$.
 - Otherwise, let $N_{-i}(m) = N(m) \setminus \{i\}$,
 $t_i(m) = \max_{\pi \in \Pi_{N_{-i}(m)}} \sum_{j \in N_{-i}(m)} v_j(\pi_j) -$

$$\sum_{j \in N_{-i}(m)} v_j(g(m)_j)$$

Intuition: i pays the loss of surplus of the other agents that it causes.

The Clarke Mechanism

- For every $i \in N$ let b_i^e be the truth revealing strategy of i : $b_i^e(v_i) = v_i$ for every $v_i \in V_i$.
- b_i^e is a weakly dominating strategy for each i .
- Revenue is calculated in that equilibrium:

$$\tilde{R} = \sum_{i=1}^n t_i(\tilde{v}, g(\tilde{v})),$$

where $\tilde{v} = (\tilde{v}_1, \dots, \tilde{v}_n)$.

- It is easy to check the following lemma:

$$\tilde{R}(v) \leq \tilde{S}(v), \quad v \in V.$$

Asymptotic Optimality

- For convenience we assume an infinite set of players $N_\infty = \{1, 2, \dots, \dots\}$.
- We consider a sequence of (similar) auctions with an increasing number of agents.
- The n^{th} auction is conducted for the set of agents $N_n = \{1, 2, \dots, n\}$
- \tilde{R}_n – the revenue obtained in the n -th auction

The Clarke Mechanism is Asymptotically Optimal

We prove:

Consider the Clarke mechanism applied to a sequence of auctions, where the n^{th} auction is conducted for the set of agents N_n , $n \geq 2$. Assume A1-A4. Then,

$$\lim_{n \rightarrow \infty} \frac{E(\tilde{R}_n)}{E(\tilde{S}_n)} = 1$$

Externalities: An Upper Bound on the Seller's Expected Revenue in Equilibrium

Consider an auction mechanism and an equilibrium profile b^e in the Bayesian game determined by this mechanism. Denote by $U_i(b^e)$ the expected utility of agent i in b^e .

We prove:

If [A4] (risk-aversion) holds, and $U_i(b^e) \geq 0$ for every $i \in N$, then the expected revenue of the seller in this equilibrium is bounded above by the expected maximal surplus.

Are the above conditions necessary for the upper bound?

Non-negativity of the agents' expected utilities in equilibrium is necessary

- One good and two risk-neutral agents (1,2). Π contains only 3 allocations: 1, 2, 0.
- One possible valuation function, w_i , for each agent i : $w_i(0) = 0$, $w_i(i) = 1$, and $w_i(3 - i) = -3$
- $M_i = \{w_i, \varphi\}$, $i = 1, 2$.
- If both players participate, $\pi = 0$ and each player pays 1.
- If only one player participates, it gets the object and pays 0.
- If both do not participate, $\pi = 0$.

It follows that:

- Declaring true valuations is the only equilibrium
- Expected revenue in equilibrium is 2, but $E(\tilde{S}) = \max_{j=0}^3 (w_1(j) + w_2(j)) = 0$.

Risk-Seeking Agents

- Risk-seeking attitude is the subject of a complementary part of our work.
- We have shown that 3rd-price (TP) auctions are preferable to 2nd price (SP) auctions when agents are risk-seeking.
- Can we obtain an expected revenue higher than the expected maximal surplus when agents are risk-seeking?

Risk-Seeking Agents

- TP for a single good with three participants, $N = \{1, 2, 3\}$.
- Independent-private-value model, and valuations uniformly distributed in the interval $[0,1]$.
- Every agent $i \in N$ uses the convex utility function $u_\alpha : R \rightarrow R$, $\alpha > 1$, where $u_\alpha(x) = x$ when $x \leq 0$, and $u_\alpha(x) = \alpha x$ for $x > 0$
- Using our previous work we show that

$$s_\alpha(v) = (1 + \sqrt{\alpha})v, \quad v \in [0, 1]$$

is an equilibrium with positive expected utility for each agent.

- In this equilibrium,

$$E_\alpha(\tilde{R}) = (1 + \sqrt{\alpha}) \cdot E(v_{[3]}) > E(\max(\hat{v}_1, \hat{v}_2, \hat{v}_3))$$

for sufficiently large α .

Summary

- Presented an upper bound on the revenue that can be obtained in multi-object auctions with arbitrary risk-averse agents
- Introduced a study of asymptotically optimal mechanism design
- Proved that the Clarke mechanism is asymptotically optimal.
- Studied the effects of externalities and risk-attitude

The upper bound: sketch of proof

Let $i \in N$. Because i is risk averse, u_i is concave. Therefore, for every $v \in V$

$$\sum_{\pi \in \Pi} u_i(v_i(\pi_i) - t_i(b^e(v), \pi))g_\pi(b^e(v)) \leq u_i\left(\sum_{\pi \in \Pi} (v_i(\pi_i) - t_i(b^e(v), \pi))g_\pi(b^e(v))\right).$$

Because i can deviate to the strategy of always not participating, its expected utility $L_i(b^e)$ is non negative. Therefore we get:

$$\int_V u_i\left(\sum_{\pi \in \Pi} (v_i(\pi_i) - t_i(b^e(v), \pi))g_\pi(b^e(v))\right) dq(v) \geq 0$$

By Jensen inequality we get:

$$u_i\left(\int_V \left(\sum_{\pi \in \Pi} (v_i(\pi_i) - t_i(b^e(v), \pi))g_\pi(b^e(v))\right) dq(v)\right) \geq 0.$$

As u_i is increasing and $u_i(0) = 0$,

$$\int_V \left(\sum_{\pi \in \Pi} (v_i(\pi_i) - t_i(b^e(v), \pi))g_\pi(b^e(v))\right) dq(v) \geq 0$$

Let \tilde{R}^i be the expected payment of i , then

$$E(\tilde{R}^i) = \int_V \sum_{\pi \in \Pi} t_i(b^e(v), \pi)g_\pi(b^e(v))dq(v).$$

We get that

$$E(\tilde{R}^i) \leq \int_V \left(\sum_{\pi \in \Pi} v_i(\pi_i)g_\pi(b^e(v))\right) dq(v).$$

As $E(\tilde{R}) = \sum_{i=1}^n E(\tilde{R}^i)$,

$$E(\tilde{R}) \leq \int_V \left(\sum_{\pi \in \Pi} \left(\sum_{i=1}^n v_i(\pi_i) \right) g_{\pi}(b^e(v)) \right) dq(v).$$

Because a convex combination of a set of numbers is less or equals the maximal number in this set,

$$E(\tilde{R}) \leq \int_V \left(\max_{\pi \in \Pi} \sum_{i=1}^n v_i(\pi_i) \right) dq(v).$$

Hence

$$E(\tilde{R}) \leq E(\tilde{S}).$$

Asymptotic optimality of Clarke: basic idea behind a (long) proof

The revenue in the n -th auction is given by

$$\tilde{R}_n = \sum_{i=1}^n t_i(\tilde{v}, g(\tilde{v})),$$

where $\tilde{v} = (\tilde{v}_1, \dots, \tilde{v}_n)$. Transform into the following representations for the revenue: For $v \in V$,

$$\tilde{R}_n(v) = \sum_{i=1}^n \left(\max_{\pi \in \Pi_{N_n \setminus \{i\}}} \sum_{j \neq i} v_j(\pi_j) - \sum_{j \neq i} v_j(g(v)) \right).$$

Note that,

$$\sum_{i=1}^n \left(\sum_{j \neq i} v_j(g(v)_j) \right) = (n-1) \tilde{S}_n(v),$$

Hence

$$\tilde{R}_n(v) = \tilde{S}_n(v) - \sum_{i=1}^n \left(\tilde{S}_n(v) - \max_{\pi \in \Pi_{N_n \setminus \{i\}}} \sum_{j \neq i} v_j(\pi_j) \right).$$

For agent i that does not receive a good in the Clarke's allocation we have:

$$\tilde{S}_n(v) = \max_{\pi \in \Pi_{N_n \setminus \{i\}}} \sum_{j \neq i} v_j(\pi_j)$$

Notice that there are at most $|G| = k$ agents who may receive a good. For any such agent i , when n is “large enough”, we get with a probability “very close” to 1, that there exists an agent i' who did not receive any good but its valuation-function is “very close” to that of i .

This yields that

$$\sum_{i=1}^n \left(\tilde{S}_n(v) - \max_{\pi \in \Pi_{N_n \setminus \{i\}}} \sum_{j \neq i} v_j(\pi_j) \right)$$

converges to 0 a.s., and therefore

$$\lim_{n \rightarrow \infty} \frac{\tilde{R}_n}{\tilde{S}_n} = 1 \quad \text{a.s.}$$

By applying the Lebesgue convergence theorem we get:

$$\lim_{n \rightarrow \infty} \frac{E(\tilde{R}_n)}{E(\tilde{S}_n)} = 1.$$

Discussion

- The Clarke mechanism is most useful for emerging Internet applications, where the number of goods in an auction is small, but the number of participants is large.
- Research has been concerned with winner determination in multi-object auctions. The related research makes the assumption that agents report their true valuations.
- In order to use the algorithms for winner determination, the Clarke mechanism has been suggested. The Clarke mechanism consists of $n + 1$ executions of the winner determination algorithm, and guarantees truth-revelation.
- Our work presents foundations to the above approach, by showing that the Clarke mechanism is indeed optimal.